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APPLICATION OF A MODEL OF A QUASIEQULIBRIUM PLASMA AND METHODS OF LIE ALGEBRA TO CALCULATIONS OF PARAMETERS OF A TRANSPORTED HIGH-CURRENT RELATIVISTIC ELECTRON BEAM

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A method for calculation of the parameters of an intense beam of charged particles transported by the magnetic field of the focusing elements is described, and the results of numerical modeling are presented.

A high-current relativistic electron beam (HREB) in a transport channel is a single-component totally ionized nonhomogeneous plasma located in the electromagnetic field of a focusing element. Application of a paraxial approximation to the description of the dynamics of particles in the beam makes it possible to use the methods of Lie algebra in calculations of the transformation operator that relates initial and final coordinates of a given particle in the phase space of canonically conjugate variables [1, 2]. The main problem that arises upon solving the problem of transport of a charged beam is that of taking into account the effect of the beam's charge on its properties. A self-consistent method for solution of of this problem for beams with static delf-fields is described in [3, 4]. In the present work, based on a plasma model and methods of Lie algebra, a solution is proposed of the problem of transport of a HREB with a time-dependent self-field. Calculations are carried out for a cylindrical beam trasported by the field of a magnetic focusing element.

The position of any particle of the beam in six-dimensional space is characterized by the vector $\xi(x, y, t, p_x, p_y, p_t)$. The independent variable z is the coordinate along the reference trajectory of beam particles, x and y denote the transverse deviation of the trajectory of the given particle from the reference trajectory, t denotes the time of particle presence in the conducting channel, and the other three quantities are the corresponding canonical momenta. The use of a reference trajectory implies description of the dynamics of beam particles within a paraxial approximation.

As is known, the most general plasma model is a kinetic description using the distribution function $g(\xi, z)$ of the statistical system of particles. Since for intense beams the interaction parameter η (the ratio of the mean potential energy of particles interaction to the mean kinetic energy of particle free motion) is much less than unity, Coulomb interaction of particles prevails over processes of particle collision [5]. This means that in calculations of the change in the distribution function one can restrict the consideration to a zeroth approximation with respect to the interaction parameter η, i.e., a change in the particle distribution in a volume of the phase space selected in the vicinity of the point ξ takes place only due to the inflow and outflowing of particles via the surface bounding the volume. In other words, $g(\xi, z)$ satisfies the Liouville equation $\dot{g}(\xi, z) = 0$ (the dot denotes the total derivative of the function with respect to the independent variable z).

Let us consider the dynamics of an arbitrary particle of the beam. Its Hamiltonian is as follows:

$$H(x, y, t, p_x, p_y, p_t; z) = -\frac{1}{c}\Big\{(p_t + q\Phi)^2 -$$

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$$- p_x - qA_x)^2 c^2 - (p_y - qA_y)^2 c^2 - m^2 c^4 \Big\}^{1/2} - qA_z \, , \tag{1}$$

where the scalar and vector potentials of the electromagnetic field Φ and A consist of two terms characterizing the action of the field of the focusing element and the field of the space charge on the particle:

$$\begin{aligned} \Phi\,(x, y; z) &= \Phi^{\text{field}}\,(x, y; z) + \Phi^{\text{beam}}\,(x, y; z)\, , \\ \mathbf{A}\,(x, y; z) &= \mathbf{A}^{\text{field}}\,(x, y; z) + \mathbf{A}^{\text{beam}}\,(x, y; z)\, . \end{aligned} \tag{2}$$

The potentials Φ^{field} and $\mathbf{A}^{\text{field}}$ are considered to be given, and their particular form is determined by the particular focusing system used for beam transport. For magnetic focusing elements (quadrupole, solenoid, etc.) $\Phi^{\text{field}} = 0$. The dependence of the potentials Φ^{beam} and $\mathbf{A}^{\text{beam}}$ on the coordinates is determined by the main transport problem of calculation of the transverse beam dimensions and angular divergence in an arbitrary cross-section of the focusing channel.

The action of the transformation operator relating the initial and final coordinates of an arbitrary particle in the beam in the phase space

$$\xi\,(z) = \mathcal{M}\xi^{\text{in}}\,(z)\, ,$$

can be considered a canonical transformation. Taking into account that canonical transformations form a simplectic group, the operator can be factorized, i.e., presented as an infinite product of Lie exponentials of an infinite sequence of certain homogeneous polynomials f_m [1, 2]. And since, firstly, the Poisson brackets that determine the Lie product are invariant with respect to canonical transformations and, secondly, ten generators of the simplectic group realize the representation of the Lie algebra, the polynomials f_m can be expressed via polynomials H_m entering the Hamiltonian.

In order to decompose the single-particle Hamiltonian (1) into polynomials we project the region of the phase space taken by the given particle at a given value of z on to the region of the phase space of the reference particle at the same value of z. This is attained by means of a canonical transformation as a result of which the temporal coordinate and the energy of the beam particle also become linked with the reference trajectory (its parameters are marked with the index 0):

$$t = T + t^0\,(z)\, , \quad x = X\, , \quad y = Y\, ;$$

$$p_t = P_T + p_t^0\, , \quad p_x = P_X\, , \quad p_y = P_Y\, .$$

For Hamiltonian (1) written in the new variables

$$\tilde{\mathcal{H}} = X, Y, T, P_X, P_Y, P_T, z) = -\frac{1}{c}\Big\{(P_T + p_t^0 + q\Phi)^2 -$$

$$- (P_X - qA_X)^2 c^2 - (P_Y - qA_Y)^2 c^2 - m^2 c^4 \Big\}^{1/2} - qA_z - \frac{P_T + p_t^0}{v_0} \tag{3}$$

(v_0 being the velocity of the reference particle) one can expand the radicand into a Taylor series in the vicinity of the reference trajectory ($X = 0$, $Y = 0$, $T = 0$, $P_X = 0$, $P_Y = 0$, $P_T = 0$) and obtain a representation of $\mathcal{H}$ as an infinite sum of homogeneous polynomials H_m.

If the explicit form of Φ^{beam} and $\mathbf{A}^{\text{beam}}$ is known and they are decomposed into polynomials over the space coordinate one can use Dragt's method [1, 2] to express polynomials f_m in terms of H_m and write the transformation operator $\mathcal{M}$ explicitly. In order to find $\mathcal{M}$, e.g., with accuracy up to terms of the fourth order, one must solve a system of three matrix differential equations, whose form is presented in [1, 2].

In order to calculate the potentials Φ^{beam} and $\mathbf{A}^{\text{beam}}$ one needs to know the evolution of the distribution function of beam particles in the process of beam transport. If we described HREB dynamics in the $6N$-dimensional phase space, then the distribution function of the microcanonical ensemble would correspond to the system under consideration (single-component totally ionized plasma consisting of N particles). But since we use the formalism of the six-dimensional phase space, our system in the equilibrium state is described by Gibbs' canonical distribution. Indeed, the phase space of each i-th particle of the system ($i = \overline{1, N}$) is projected onto the phase space of the reference particle. Then, the field of the space charge does not act on the reference particle, therefore its kinetic energy remains constant. Each i-th particle is subjected to the action of the field of the space charge, and this results in a change in its potential and kinetic energies. However, the mean energy of the system under consideration is conserved. As a result of interactions between plasma particles, the transverse four-dimensional volume occupied by the system changes. Therefore, the transverse emittance of the beam should change, and a change in the system takes place. In order to calculate this variation, one must solve Vlasov's equation combined with Maxwell's system of equations.

However, an alternate way exists. Each i-th particle of an arbitrary cross-section of the beam is subjected to the field of the space charge created by other particles and described by potentials $\Phi^{\text{beam}}(x, y; z)$ and $\mathbf{A}^{\text{beam}}(x, y; z)$. Its form can be considered to be constant along z within the limits of a certain length l exceeding the Debye length but less than the free-path length for electrons in the plasma. Therefore, on each elementary portion l of the transport route, particles moving along the z-axis experience the action of the time-invariant electromagnetic field. At the same time, along the entire route the distribution function satisfies the Liouville equation. Therefore, in the process of transport the beam as a statistical system transforms successively from one equilibrium state to another. It is evident that the volume occupied by the particles plays here the part of a slowly varying parameter of this adiabatic process. Thus, the distribution function of the quasiequilibrium plasma under consideration is functionally Gibbs' distribution along the entire transport channel and changes solely its shape on each elementary portion l.

Including in the consideration the magnetostatic field of the focusing element does not disrupt the equilibrium in the system since its mean energy does not change.

For a plasma in a state of statistical equilibrium, the electric and magnetic fields are not interrelated [7]. Therefore, to calculate $\mathbf{A}^{\text{beam}}$ on each elementary portion l, one can use the quasistationary approximation [8]:

$$\mathbf{A}_X^{\text{beam}}(X, Y) = 0\,, \quad \mathbf{A}_Y^{\text{beam}}(X, Y) = 0\,, \quad \mathbf{A}_z^{\text{beam}}(X, Y) = \frac{v_0}{c^2}\,\Phi^{\text{beam}}(X, Y)\,.$$

Thus, the self-electric field of the beam $\mathbf{E}^{\text{beam}} = -\text{grad}\,\Phi^{\text{beam}}$ appears to be strictly transverse with respect to the reference trajectory, and its self-magnetic field $\mathbf{B}^{\text{beam}} = \text{rot}\,\mathbf{A}^{\text{beam}}$ is azimuthal.

In order to calculate Φ^{beam} and $\mathbf{A}^{\text{beam}}$ on each elementary portion l, one must know the distribution function of particles over the transverse spatial coordinates and corresponding momenta. Usually, in problems of beam transport and in numerical modeling of this process it is assumed that the transverse coordinates of particles have a Gaussian distribution, whereas the distribution of conjugate momenta is described by Maxwell's statistics [3, 4]:

$$g(X, Y, P_X, P_Y) = \frac{1}{2\pi\sqrt{\sigma_X \sigma_Y}}\,\frac{1}{2\pi\sqrt{\lambda_X \lambda_Y}} \times$$

$$\times \exp\left[-\frac{1}{2}\frac{(X-\alpha)^2}{\sigma_X} - \frac{1}{2}\frac{(Y-\beta)^2}{\sigma_Y} - \frac{1}{2}\frac{(P_X-\gamma)^2}{\lambda_X} - \frac{1}{2}\frac{(P_Y-\delta)^2}{\lambda_Y}\right], \qquad (4)$$

where α, β, γ, and δ are the expected values of the random quantities X, Y, P_X, P_Y; σ_X, σ_Y, λ_X, λ_Y are variances of these quantities.

According to (4), the expression for the volume density of the beam charge $\rho(X, Y)$ on each elementary portion l is as follows:

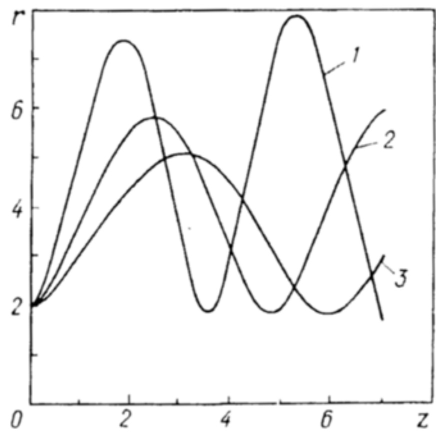

Fig. 1. Variations in transverse dimensions of electron beam moving in homogeneous magnetic field: 1) B_0 = 30 G; 2) 40; 3) 50. z, r, m.

$$\rho (X, Y) = \frac{I}{v_0} \frac{1}{2\pi \sqrt{\sigma_X \sigma_Y}} \exp \left[-\frac{1}{2} \frac{(X-\alpha)^2}{\sigma_X} - \frac{1}{2} \frac{(Y-\beta)^2}{\sigma_Y} \right] \quad (5)$$

where I is the current of the beam being transported.

Then the value of Φ^{beam} on a particular transport portion is calculated using Green's function as a solution of the two-dimensional Dirichlet boundary-value problem for Poisson's equation:

$$\varphi^{\text{beam}} (x_0, y_0) = \iint\limits_{(xy)} dx\, dy \frac{\rho}{\varepsilon_0} \frac{1}{2\pi} \ln \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \quad (6)$$

where ε_0 is the dielectric constant of vacuum.

The polynomial expansion of $\Phi^{\text{beam}}(X, Y)$ on each elementary portion l can be performed using the method of least squares. In order to do this, a transverse grid is superimposed on the transverse cross-section of the beam, the values of Φ^{beam} are calculated at the nodes of the grid according to (6), and then the tabulated function is approximated by homogeneous polynomials. Correspondingly, the coefficients of the polynomial expansion of $A^{\text{beam}}(X, Y)$ will differ by the v_0/c^2 multiplier.

Thus, the scheme for calculation of the basic parameters of the HREB in the process of transport looks as follows. The initial cross-section of the beam is represented by probe particles whose distribution function over the transverse coordinates and momenta is determined by the expression (4). From (6) we find numerically the values of Φ^{beam} at the nodes of the grid superimposed on the cross-section by calculating the integral, e.g., by Gauss' method over a hyperrectangle. Then functions $\Phi^{\text{beam}}(X, Y)$ and $A^{\text{beam}}(X, Y)$ are approximated by homogeneous polynomials up to the fourth order and substituted into the Hamiltonian (3), and we find expressions for H_m. Solving numerically the system of three matrix differential equations, e.g., by the Runge–Kutta–Merson method, we obtain the explicit form of the transformation operator $\mathcal{M}$. With its help we find the coordinates and momenta of the probe particles at the end of the first elementary portion. By calculating the mean values and variances of the transverse coordinates and momenta of probe particles, and their energy spectrum, we obtain the initial data on the probe particles for calculations in the next transport portion. By repeating the procedure of evaluation of the potentials of the field of the space charge of the beam and the transformation operator, we carry out calculations in succeeding steps.

Based on the afore-described algorithm, the TRLIE software for a PC was developed for calculations of HREB parameters in a transport channel with magnetic focusing elements.

By way of example let us consider the change in the beam radius under strong focusing in a solenoid (plasma cylinder of radius r_0 in a constant magnetic field B_0). Results of calculations using the TRLIE program for various values of B_0 are presented in Fig. 1. The calculations were carried out for a HREB with a kinetic energy

TABLE 1. Characteristics of Surface Waves in the Beam and Plasma

E, MeV	k, m^{-1}	w, sec^{-1}	ω, sec^{-1}
1	1.70	$2.77 \cdot 10^9$	$2.51 \cdot 10^9$
1.5	1.28	$2.88 \cdot 10^9$	$2.72 \cdot 10^9$
2	1.03	$2.93 \cdot 10^9$	$2.80 \cdot 10^9$

of 10 keV, a current of 10 kA, and initial radius of 10 cm, and an initial scatter in transverse velocities equal to 1% of the longitudinal velocity of particles. The cross-section of the beam was represented by 100 probe particles.

The obtained oscillations of the boundary of the beam can be identified with surface oscillations of a plasma spatially confined by an external longitudinal magnetic field. The dispersion equation for axisymmetric modes yields the following expression for the spectrum of those longwave oscillations (the frequency ω does not exceed the plasma frequency ω_L) [7]:

$$\omega^2 = k^2 \frac{\omega_L^2 r_0^2}{2} \ln \frac{1}{kr_0}. \tag{7}$$

Table 1 presents values of the wavenumber k and frequency ω of surface waves obtained using the TRLIE software for various values of induction of the focusing magnetic field and values of ω calculated from (7).

Here we should point out the following circumstances. Firstly, frequencies ω and w are calculated in different frames: ω – in the laboratory frame and w in a frame moving along with the beam. Multiplying w by the Lorentz factor γ we obtain the values of frequencies of the surface oscillations of the beam in the laboratory frame. Secondly, since oscillations of the beam boundary manifest themselves as oscillations of the radius of the beam cross-section, it seems natural to consider their phase velocity to be equal to the velocity of the reference particle of the beam. Thirdly, we take the minimum value of the HREB radius r_0 as a parameter of plasma inhomogeneity. According to the data from Table 1, the modeling results agree with the known facts of plasma physics.

The proposed model of quasiequilibrium plasma and methods of Lie algebra make it possible to calculate the basic parameters of a transported HREB. In addition, it helps to relate phenomena that take place in a beam of charged particles moving in an external electromagnetic field to processes that take place in spatially confined single-component collisionless plasma.

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REFERENCES

1. A. J. Drag and J. M. Finn, J. Math. Phys., **17**, 2215-2227 (1976).
2. A. J. Drag and E. Forest, J. Math. Phys., **24**, 2734-2744 (1983).
3. R. D. Ryne, A. J. Dragt, in: Proc. of the 1987 IEEE PAG., New York (1987), pp. 1063-1065.
4. R. D. Ryne, in: AIP Conf. Proc., No. 177, New York (1987), pp. 265-274.
5. A. F. Aleksandrov, L. S. Bogdankevich, and A. A. Rukhadze, Oscillations and Waves in Plasma Media [in Russian], Moscow (1990).
6. A. F. Aleksandrov, L. S. Bogdankevich, and A. A. Rukhadze, Principles of Plasma Electrodynamics [in Russian], Moscow (1988).
7. I. M. Kapchinskii, Theory of Linear Resonance Accelerators [in Russian], Moscow (1982).